TOWARD A CATEGORICAL DONALDSON-UHLENBECK-YAU CORRESPONDENCE

This a joint project with Fabian Haiden, Maxim Konstevich and Pranav Pandit, which introduces and explores the notion of Kähler metrics in the context of categorical noncommutative algebraic geometry. The introduction of differential geometric features into categorical geometry appears completely novel, and will have far reaching applications in algebraic and symplectic geometry. This project constitutes a synthesis of ideas, questions and techniques coming from string theory, gauge theory, symplectic geometry, calibrated geometry, geometric invariant theory and partial differential equations.

A cornerstone of the profound relationship between differential geometry and algebraic geometry is the theory of special metrics, and in particular, the Donaldson-Uhlenbeck-Yau (DUY) theorem. A special case of this theorem asserts that, on a smooth projective complex variety X, there is a correspondence between polystable bundles and Hermitian-Yang-Mills connections. The former is a purely algebro-geometric notion defined using Mumford's GIT, while the latter is a purely differential geometric construct, defined using the Kähler metric on X. The essential content of the DUY theorem is the assertion that a holomorphic bundle is polystable iff it admits a unitary connection compatible with the complex structure (or equivalently, a hermitian metric), that *minimizes* a certain functional.

The idea of studying structures by finding representatives of that structure that extremize a certain quantity is extremely robust, and consequently ubiquitous in mathematics. It is the basis for some of the deepest theorems in mathematics – it constitutes the basis for Hodge theory and Hamilton's Ricci flow program, for example. The project described in this proposal derives from the observation that the key features of the Donaldson-Uhlenbeck-Yau extremization problem are sufficiently robust to allow one to extend the DUY theorem to the context of categorical noncommutative geometry. In the categorical realm, where point set definitions cannot be made, our formulation of the categorical DUY correspondence will serve as a proxy for the notion of a categorical Kähler metric. Here is a summary of the main aspects and stages of this project:

A) Formulate the notion of a Kähler/DUY category. Show that categories of A-branes and B-branes can be enhanced to DUY categories.

B) Prove that a Kähler/DUY structure gives rise to (i) a Kähler metric on moduli spaces of polystable objects and (ii) an underlying Bridgeland stability structure on the category.

C) Prove a DUY theorem for Kähler/DUY categories.

D) Use B) and C) to construct Bridgeland stability structures on various classes of categories, and to describe stable objects in terms of "metric structures". The emphasis is on categories of *symplectic* origin, including Fukaya categories with coefficients in a perverse Schober.

E) Investigate categorical Donaldson invariants and then the connection with rationality.

The notion of a Kähler or DUY category can be seen as a mathematical axiomization of the essential features of stable D-branes in N = (2,2) supersymmetric field theories in physics. Indeed, our thinking is very strongly influenced by Douglas' seminal work on π-stability of D-branes. Kontsevich observed that branes are most naturally organized into dg-categories, and developed a theory of categorical noncommutative geometry based on this observation. Bridgeland axiomatized certain features of π-stability in the notion of a stability condition on a triangulated dg-category. However, Bridgeland's approach makes no mention of the differential geometric structures and PDE that inform Douglas' physical reasoning. Our work aims to revisit Bridgeland stability conditions and reinterpret them in a manner that is more directly faithful to the physics.

The quintessential example of a noncommutative space is the derived category of coherent sheaves (the B-model, or the category of B-branes) on a complex variety. In this situation, the notion of a Kähler category generalizes the usual notion of Kähler metric in the sense that any Kähler metric on the variety gives rise to a categorical Kähler metric on the category of B-branes. Applied to the B-model, our framework subsumes the usual DUY correspondence for bundles by extending it to complexes of vector bundles, and also subsumes the notions of stability described by the *deformed* HYM equations studied by Conan Leung, Yau, Jacob-Yau and others. The deformed HYM equations arise in string theory in the description of stable B-branes.

The philosophy of categorical noncommutative geometry derives from the idea that (higher) categories provide a powerful unified framework in which to study geometric phenomena arising from very disparate contexts, and to understand the surprising relationships between these geometries predicted by dualities in string theory. The novelty and the power of the theory of categorical Kähler metrics is that it allows us to apply the ideas of the DUY correspondence, which originally pertains to bundles, to new contexts; indeed the main application we have in mind is to symplectic geometry and theory theory of special Lagrangian manifolds and spectral networks. The existence of DUY structures on Fukaya categories will allow us to prove the existence of Bridgeland stability structures associated to holomorphic volume forms, and the DUY correspondence in this context will assert that an object of the Fukaya category is polystable if and only if it admits a special Lagrangian representative. In other words, in the A-model, a "metric" on an object of the Fukaya category is the choice of an explicit Lagrangian representative of that object, and the "metric" is HYM iff the Lagrangian is *special*.

As mentioned above, we envisage that our definition of Kähler/DUY categories will allow us to develop a theory of categorical Donaldson invariants, analogous to Donaldson's theory for bundles. We expect that this theory will have far reaching applications to questions of rationality in algebraic geometry.

1. The Donaldson-Uhlenbeck-Yau correspondence and Categorical Kähler geometry

All of our constructions, statements and theorems will have two avatars: an Archimedean one, and a non-Archimedean one. To simplify the discussion, and to avoid having to repeat statement twice, we will suppress the context (archimedean/non-archimedean) in our notation: throughout what follows we will assume either that $K = \mathbb{C}$, or that K is a non-Archimedean field with ring of integers $\mathcal{O}_K$. Many of

the statements are given only in the non-Archimdean case, although they can be suitably reinterpreted in the case $K = \mathbb{C}$ if $\mathcal{O}_K$ is given the appropriate meaning.

To fix our situation, we begin by defining what we mean by a noncommutative space:

Definition 1.1. An **nc**-space is a presentable, stable K-linear ∞-category.

1.1. Kempf-Ness and Donaldson-Uhlenbeck-Yau. The Donaldson-Uhlenbeck-Yau correspondence can be fruitfully thought of as an infinite-dimensional analogue of the Kempf-Ness theorem. This point of view was introduced and fruitfully exploited by Atiyah-Bott and Donaldson. The Kempf-Ness theorem, in turn, is a non-linear analogue of a familiar idea from elementary linear algebra. In the table below, we summarize the main analogies between various structures appearing in the Kempf-Ness/Donaldson-Uhlenbeck-Yau theorem on the one hand, and in the theory of categorical Kähler geometry that we propose to develop on the other. Recall that, in the DUY situation, the space X in the table is the space of unitary connections of type (1,1) on a smooth hermitian vector bundle, K is the unitary gauge group, G is the complex gauge group, L is Quillen's determinant line bundle, and the moment map is given by curvature.

1.2. Metrized branes. Let $\mathcal{X}$ be an **nc**-space. In order to formulate a categorical analogue of the Donaldson-Uhlenbeck-Yau correspondence, it is necessary to introduce extra structure on $\mathcal{X}$. The notion of a metric on an object of $\mathcal{X}$ is necessary in order to be able to even speak of the main entities appearing on the differential geometric side of the Donaldson-Uhlenbeck-Yau correspondence, namely, Hermitian-Yang-Mills metrics.

Definition 1.2. Let $\mathcal{X}$ be an **nc**-space, ie, a K-linear presentable ∞-category. A metrization scheme for $\mathcal{X}$ consists of a $\mathcal{O}_K$-linear ∞-category $\mathcal{X}^\circ$ and a functor $\pi : \mathcal{X}^\circ \to \mathcal{X}$ such that the induced functor $\mathcal{X} \otimes_{O_K} K \to \mathcal{X}$ is an equivanlence.

GIT & symplectic reduction	Categorical Kähler geometry
Variety (X, I) with G-action $\leftrightarrow [X/G]$	K-linear stable ∞-category $\mathcal{X}$
Linearization of G action	Categorical line bundle $\mathcal{L} : \Omega\lvert s_\bullet \mathcal{X} \rvert \to \mathrm{Pic}(\mathrm{Mod}_k)$
Mumford stability	Bridgeland stability
Linearization $K \to \mathrm{Aut}(L, h)$ for (L, h) hermitian line bundle	Metrized categorical line bundle $\mathcal{L} : \Omega\lvert s_\bullet \mathcal{X}^\circ \rvert \to \mathrm{Pic}(\mathrm{Mod}_{\mathcal{O}_K})$
Comoment map $\mathfrak{K} \to C^\infty(X)$	Central charge $Z : K_0(\mathcal{X}) \to \mathbb{C}$
Zeroes of moment map $\Phi^{-1}(0) \subset X$	Full subcategory $\mathcal{X}^{\mathrm{HYM}} \subset \mathcal{X}^\circ$ of HYM metrized objects
Negative Gradient flow of $\phi := (\Phi, \Phi)$	Flow $\tilde{\mathcal{F}}$ on moduli of objects in $\mathcal{M}_{\mathcal{X}^\circ}$
Metrized vector bundles on (X, I)	$\mathcal{X}^0$, such that $\mathcal{X}^\circ \otimes_{\mathcal{O}_K} K \simeq \mathcal{X}$
Symmetric space G/K	$\mathfrak{Met}(E) := \mathrm{hofib}_E(\mathcal{X}^\circ \to \mathcal{X})$
Pullback of ϕ via $G/K \to X/K$; $g \mapsto [gx]$	Mass M
Pullback of $-\mathrm{grad}\phi$ via $G/K \to X/K$	Flow $\mathcal{F}$ on $\mathrm{Met}(E)$
Kempf-Ness function $\psi_v : G/K \to \mathbb{R}$; $\psi_v(x) = \log \lVert gx^2 \rVert$	complexified Kähler potential $S_E : \mathrm{Met}(E) \to \mathbb{C}$
$d\psi_v(g) = 0 \Leftrightarrow [gx] \in \Phi^{-1}(0)$	$E^\circ \in \mathcal{X}_\theta^{\mathrm{HYM}} \Leftrightarrow \delta S_E^\theta(E^\circ) = 0$ where $S_E^\theta := \mathrm{Im}(e^{i\theta} S_E)$
KEMPF-NESS THEOREM $\Phi^{-1}(0)/K \simeq X^{ps}/G$	CATEGORICAL DONALDSON-UHLENBECK-YAU. The functor $\mathcal{X}^{\mathrm{HYM}} \to \mathcal{X}$ induced by π induces an equivalence $\mathcal{X}_\theta^{\mathrm{HYM}} \simeq \mathcal{X}_\theta^{ps}$.

1.3. Categorical line bundles and comoment maps. In order to define categorical analogues of a linearization of G-action of a variety, we need to introduce the notion of a line bundle on category. Intuitively, this an assignment of a line $\mathcal{L}_E$ to every object of the category, and for every exact triangle $E' \to E \to E''$ an isomorphism $\mathcal{L}_E \simeq \mathcal{L}_{E'} \otimes \mathcal{L}_{E''}$. More generally, one can define the notion of a line bundle on a category with values in any picard spectrum; our example corresponds to special case of the Picard spectrum of invertible K-modules. The formal defintiion is a follows:

Definition 1.3. Let $\mathcal{A}$ be a Picard spectrum, i.e., a grouplike $\mathbb{E}_\infty$ ring, and let $\mathcal{X}$ be a stable ∞-category. An $\mathcal{A}$-valued line bundle on $\mathcal{X}$ is an $\mathbb{E}_1$-ring map

$$\Omega\lvert s_\bullet \mathcal{X} \rvert \to \mathcal{A}$$

Given a variant of a line bundle on a category, a major goal of the project is to define something analogous to a Bridgeland stability condition on $\mathcal{X}$ using a variant of Mumford's ideas. The idea is to look at the action of $\mathrm{Aut}(E)$ on $\mathcal{L}_E$ in families and give characterizations of semistable, polystable, etc, in the spirit of Mumford's GIT.

Similarly, we would like to define an analogue of the comoment map for categories.

1.4. Donaldson-Uhlenbeck-Yau correspondence for Kähler categories.

Definition 1.4. A Kähler structure on a **nc**-space $\mathcal{X}$ consists of
  (1) A metrization scheme $\pi : \mathcal{X}^\circ \to \mathcal{X}$, $\mathcal{X}^\circ \otimes_{\mathcal{O}_K} K \simeq \mathcal{X}$ for $\mathcal{X}$.
  (2) A polarization of $\mathcal{X}$: an S^1-family of lines bundles $\mathcal{L}_\theta$ on $\mathcal{X}$, $\theta \in S^1$. This family is determined by a pair of line bundles $\mathcal{L}_\mathbb{R}$ and $\mathcal{L}_\mathbb{I}$.
  (3) A comoment map for $\mathcal{X}^\circ$, given by an S^1-family of metrized line bundles $\mathcal{L}^\circ$ on $\mathcal{X}^\circ$.
  (4) A compatibility condition between $\mathcal{L}^\circ$ and $\mathcal{L}$.

Conjecture 1.5. *Let $\mathcal{X}$ be a smooth and proper **nc**-space equipped with a Kähler structure. Then the Donaldson-Uhlenbeck-Yau corresondence holds for $\mathcal{X}$, i.e., the functor $\pi : \mathcal{X}^\circ \to \mathcal{X}$ induces an equivalence $\mathcal{X}^{\mathrm{HYM}} \to \mathcal{X}^{ps}$ where $\mathcal{X}^{ps} \subset \mathcal{X}$ is the full subcategory of polysable objects for the linearization of $\mathcal{X}$, and $\mathcal{X}^{\mathrm{HYM}} \subset \mathcal{X}^\circ$ is the full subcategory of Hermitian-Yang-Mills objects.*

1.5. Donaldson-Uhlenbeck-Yau categories. For practical applications, it is useful to consider various shadows of a Kähler structure on a category. In this section we introduce the notion of a Donaldson-Uhlenbeck-Yau structure on a category. We expect any Kähler structure to induce a Donaldson-Uhlenbeck-Yau structure. Furthermore, in the presence of a Donaldson-Uhlenbeck-Yau (DUY) structure, we expect the Donaldson-Uhlenbeck-Yau correspondence to hold. In the absence of a complete definition of a Kähler structure on a category, we propose that one may treat a DUY structure on a category as a proxy for the notion of a Kähler metric on a category.

Definition 1.6. Let $\mathcal{X}$ be an **nc**-space over k.
  (1) A *pre-metrization scheme* for $\mathcal{X}$ consists of an **nc**-space $\mathcal{X}^\circ$ and a functor $\mathrm{oblv} : \mathcal{X}^\circ \to \mathcal{X}$.
  (2) Let E be an object of $\mathcal{X}$. The *space of metrics* $\mathrm{Met}(E)$ on E relative to $(\mathcal{X}^\circ, \mathrm{oblv})$ is defined by the homotopy pullback square:

$$\begin{array}{ccc} \mathrm{Met}(E) & \longrightarrow & \mathcal{X}^\circ \\ \downarrow & & \downarrow {\scriptstyle \mathrm{oblv}} \\ \{*\} & \xrightarrow{\ E\ } & \mathcal{X} \end{array}$$

  Explicitly, a point in $\mathrm{Met}(E)$ is specified by giving an object $E^\circ \in \mathcal{X}^\circ$ and an equivalence $h : \mathrm{oblv}(E^\circ) \to E$.
  (3) A *metrization scheme* for $\mathcal{X}$ is a pre-metrization scheme, together with the structure of a Hadamard metric space (i.e., CAT(0) space, or space of non-postive curvature) on the coarse moduli space of the topological stack $\mathrm{Met}(E)$ for every E in $\mathcal{X}$.
  (4) A *good metrization scheme* for $\mathcal{X}$ consists of a metrization scheme, together with a compactification $\overline{\mathrm{Met}}(E)$ of $\mathrm{Met}(E)$ for each $E \in \mathcal{X}$, such that the points at infinity correspond to flags

$$E_1^\circ \to E_2^\circ \to ... \to E_r^\circ$$

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in $\mathcal{X}^{\circ}$ with $\mathrm{oblv}(E_r^{\circ}) \simeq E$.

We are now ready to introduce the structure necessary to formulate the categorical Donaldson-Uhlenbeck-Yau correspondence.

Definition 1.7. Let $\mathcal{X}$ be an **nc**-space. Then $\mathcal{X}$ is equipped with *Donaldson-Uhlenbeck-Yau* data, if we are given:

(Z) A group homomorphism $Z : K_0(\mathcal{X}) \to \mathbb{C}$ called the *central charge*.

(H) A good metrization scheme on $\mathcal{X}$ as in Definition 1.6.

(S) A complex valued function S_E on $\mathrm{Met}(E)$ for every $E \in \mathcal{X}$, called the *complexified Kähler potential*.

(M) A function $M_E : \mathrm{Met}(E) \to \mathbb{R}_{\geq 0}$ for each $E \in \mathcal{X}$ called the *mass*.

(F) For every E, a one-parameter group $\mathcal{F}_E : (\mathbb{R}_{\geq 0}, +) \to \mathrm{Aut}(\mathrm{Met}(E))$ of continuous automorphisms of $\mathrm{Met}(E)$ called *the flow*.

(A) A pair of real valued functions Amp_+^E and Amp_-^E on $\mathrm{Met}(E)$.

From now on, we will suppress the subscript/superscript E in our notation, whenever E is clear from context, or when quantifying over all E.

Definition 1.8. Let $\mathcal{X}$ be an **nc**-space equipped with Donaldson-Uhlenbeck-Yau data. Then $\mathcal{X}$ is a *Donaldson-Uhlenbeck-Yau category* if the following axioms are satisfied:

(A1) The *BPS inequality* $M_E(E^{\circ}, h) \geq |Z(E)|$ holds for all E and for all $(E^{\circ}, h : \mathrm{oblv}(E^{\circ}) \to E) \in \mathrm{Met}(E)$.

(A2) The real valued function $S_E^{\theta} := \mathrm{Im}(e^{i\theta} S_E)$ is convex along geodesics for every $\theta \in \mathbb{R}$. At a critical point of S_E^{θ}, the flow acts on metrics via the rescaling action of θ.

(A3) The mass and flow extend continuously to the compactification $\overline{\mathrm{Met}}(E)$.

(A4) The mass decreases along the flow, i.e., $M_E(\mathcal{F}_E(t)x) \leq M_E(x)$ for all $x \in \mathrm{Met}(E)$ and all $t \geq 0$.

(A5) The complexified Kähler potential is additive with respect to direct sums: $S_{E \oplus F}((E^{\circ}, h_E)) \oplus (F^{\circ}, h_F)) = S_E((E^{\circ}, h_E)) + S_F((F^{\circ}, h_F))$.

(A6) The amplitude functions satisfy:

$$Amp_+((E^{\circ}, h_E)) \oplus (F^{\circ}, h_F) = max\{Amp_+(E^{\circ}, h_E), Amp_+(F^{\circ}, h_F)\}$$

$$Amp_-((E^{\circ}, h_E)) \oplus (F^{\circ}, h_F) = min\{Amp_-(E^{\circ}, h_E), Amp_-(F^{\circ}, h_F)\}$$

(A7) There are various compatibilities between the rescaling action on Met, the suspension functor, and the flow, which we will not spell out explicitly here. There are also compatibilities between these actions and the various functions above; the most fundamental of these is that under the rescaling action on metrics, S_E transforms by rescaling by $Z(E)$.

The structure of a Donaldson-Uhlenbeck-Yau category allows us to formulate the notion of a polystable object in $\mathcal{X}$, which is the main entity appearing on the algebro-geometric side of the Donaldson-Uhlenbeck-Yau correspondence.

Definition 1.9. Let $\mathcal{X}$ be a Donaldson-Uhlenbeck-Yau category. For $\theta \in \mathbb{R}$, let $\mathrm{Met}(E)_{\theta \pm \pi} := \{(E^{\circ}, h) \in \mathrm{Met}(E) | \theta - \pi < Amp_-(E^{\circ}, h_E) \leq Amp_+(E^{\circ}, h_E) \leq \theta + \pi\}$. Then an object E in $\mathcal{X}$ is

- *polystable of slope* $\theta \in \mathbb{R}$ if $\theta = arg(Z(E))$ and the function $S_E^{\theta} := \mathrm{Im}(e^{i\theta} S_E)$ attains a minimum on $\mathrm{Met}(E)_{\theta \pm \pi}$.

- *semistable of slope θ* if $\theta = arg(Z(E))$ and S_E^θ is bounded below on $\mathrm{Met}(E)_{\theta \pm \pi}$. We denote by $\mathcal{X}^\theta$ the full subcategory of $\mathcal{X}$ consisting of semistable objects of slope θ.

The following conjecture is one of the main motivations for introducing the notion of a Donaldson-Uhlenbeck-Yau category.

Conjecture 1.10. *Let $\mathcal{X}$ be a Donaldson-Uhlenbeck-Yau category with central charge Z. Let $\mathcal{X}^\theta$ denote the subcategory of semistable objects of slope θ in $\mathcal{X}$, as defined in Definition 1.9. Then $(\mathcal{X}, Z, \{\mathcal{X}^\theta\}_{\theta \in \mathbb{R}})$ defines Bridgeland stability structure on $\mathcal{X}$.*

Definition 1.11. Let $\mathcal{X}$ be a Donaldson-Uhlenbeck-Yau category, with category of metrized objects $\mathcal{X}^\circ$. An object E° in $\mathcal{X}^\circ$ is *Hermitian-Yang-Mills of slope θ* if it is a critical point of the functional S_E^θ. We say that an object $E \in \mathcal{X}$ *admits a Hermitian-Yang-Mills metric of slope θ* if there is a Hermitian-Yang-Mills object $E^\circ \in \mathcal{X}^\circ$ of slope θ and an equivalence $\mathrm{oblv}(E^\circ) \simeq E$.

Conjecture 1.12. *Let $\mathcal{X}$ be a Donaldson-Uhlenbeck-Yau category. Then an object $E \in \mathcal{X}$ is polystable of slope θ if and only if it admits a Hermitian-Yang-Mills metric of slope θ. This induces an homeomorphism between the moduli space of Hermitian-Yang-Mills objects in $\mathcal{X}^\circ$ of slope θ and the moduli space of semistable objects of slope θ in $\mathcal{X}$.*

Donaldson-Uhlenbeck-Yau correspondence	$M^{\mathrm{Brid},\theta} \overset{\mathrm{homeo}}{\cong} M^{\mathrm{HYM},\theta}$ mod space of Bridgeland stab. mod space of HYM - Hermitian-Yang-Mills

2. Avatars of the axiomatic framework

There are two main geometric contexts that inform our theory of DUY categories: the A-model (symplectic geometry) and the B-model (algebraic geometry). We are especially interested in the relationshiop between the A-model and the B-model predicted by homological mirror symmetry (HMS). The HMS phenomenon is best understood in the context of a degenerating family of varieties approaching a cusp in the moduli space of CFTs. This situation is most naturally described, following Kontsevich-Soibelman, by working with categories linear over a non-Archimedean field. Finally, in order to get a better handle on the ideas, it is useful to keep in mind two "toy examples", namely categories of representations of quivers, and what may be called the "local example".

To summarize we have two underlying contexts we wish to consider:
- Archimedean, with $K = \mathbb{C}$.
- non-Archimedean (e.g., $K = \mathbb{C}((t^{\mathbb{R}}))$ is a Novikov field)

and there are at least the following four classes of noncommutative spaces, to which we will apply the theory of DUY categories:
- the A-model (Fukaya categories with coefficients in a perverse Schober)
- the B-model (derived categories of coherent sheaves)

- derived categories of representations of a quiver
- The "local example": a category $\mathcal{C}$ over a non-archimedean field K (the **nc**-space), a category $\tilde{\mathcal{C}}$ linear over $\mathcal{O}_K$, together with (i) an isomorphism $\tilde{\mathcal{C}} \otimes_{\mathcal{O}_K} K \simeq \mathcal{C}$ and (ii) a Bridegeland stability structure on the special fiber $\tilde{\mathcal{C}} \otimes_{\mathcal{O}_K} k$.

Note that the fourth of these classes of categories is only defined in the non-Archimedean context, and so are Fukaya categories in most cases.

In the table below, we give explicit formulas describing DUY structures on each of these classes of categories. The formulas are for the mass, Kähler potential, flow, etc.

Local Example	Metrized Objects	Flow $\mathcal{F}$	Kähler Potential S	Mass M
K non-arch $\mathcal{X}$ linear $/K$ $\mathcal{X}^0/\mathcal{O}_K$ model: $\mathcal{X}^0 \otimes_{\mathcal{O}_K} K \simeq \mathcal{X}$ + stability on special fiber $\mathcal{X}^0 \otimes_{\mathcal{O}_K} k$	For $E \in \mathcal{X}$ $\mathrm{Met}(E) =$ $\mathrm{hofib}_E(\mathcal{X}^0 \to \mathcal{X})$	rescale differential	$S(E_1, E_2, \alpha) =$ $\mathcal{J}(Cone(E_1, E_2, \alpha))$ $\mathcal{J}$ - torsion Amp jumps	defined using central charge on central fiber
A - model C $\downarrow C_Y$ $L \quad (B, w)$ $Z : K^0(C_X) \to K_X$ $\mathcal{X} = \mathrm{Fuk}(X, C_X)$	Objects of $\mathcal{X}^0$: Singular Lag + bound. cochain + C_X-decoration	$Z = \int_L \Omega$ SL= graph of df flow $\mathcal{F} = \mathrm{Arg}_{loc}$	$dS =$ $= \int_L \Omega f$ $f_{\mathcal{F}} = \mathrm{Arg}$ loc $\delta L =$ graph of f	$M =$ $= \int_L |\Omega_L|$
B - model (X, ω, I) Kähler	Some objects complexes of hermitian vector bundles	$\mathcal{K} = e^{i\omega} \Gamma_X$ $Z = \int Tr(e^F \mathcal{K})$ $\mathrm{Arg}{-} = \inf \mathrm{Arg}$ $\mathrm{Arg}{+} = \sup \mathrm{Arg}$ $\mathcal{F} = (\delta h)h^{-1} =$ $\mathrm{Arg}(e^F \mathcal{K}) = \mathrm{Arg}_{local}$	$\delta S =$ $\int \mathrm{Tr}\left((\delta h)h^{-1}e^F\mathcal{K}\right)$	$M =$ $= \int \mathrm{Tr} |e^F\mathcal{K}|$

Quivers	$E = \bigoplus_{v,j} E_v^j$ v - vertices $j \in \mathbb{Z}$	$\mathcal{F} = \pi \operatorname{Deg}$ $- \arg(Z + B_0)$	$S =$ $\sum_{v,j}(-1)^j Z_v \log h_v^j$ $- \sum_{\alpha,j} \operatorname{Tr}(T_\alpha^{j+} T_\alpha^j)$ $- \sum_{\alpha,j} \operatorname{Tr}[d^+, d].$	$Mass =$ $\operatorname{Tr}(Z + B_0)$ $- \operatorname{Tr}(\Delta)$

Example 2.1. (**Quiver representations: a toy example**)
Let Q be a quiver with vertex set V and set of arrows A. Let ρ be a metrized object of the derived category of representations of Q; explicitly, this means that we are given:

- for each vertex $v \in V$ a chain complex (E_v^j, d_v^j) of K-vector spaces, $j \in \mathbb{Z}$, $d_v^j : E_v^j \to E_v^{j+1}$
- for each $v \in V$ and $j \in \mathbb{Z}$ a hermitian metric h_v^j on E_v^j
- for each $\alpha \in A$ and $j \in \mathbb{Z}$ a map of K vector spaces $T_\alpha^j : E_{t(\alpha)}^j \to E_{h(\alpha)}^j$, where $t(\alpha)$ (resp. $h(\alpha)$) is the tail (resp. head) of the arrow α.

To simplify the notation, we introduce the following definitions:

$$E = \bigoplus_{v,j} E_v^j, \quad d = \sum_{v,j} d_v^j, \quad T_\alpha = \sum_j T_\alpha^j$$

$$h = \oplus h_i, \quad d^2 = 0, \quad T_\alpha \cdot d = d \cdot T_\alpha$$

Let $Z = \sum_{v_j} Z_v \, pr_{v_j}$ and $B_0 = [d^+, d] + \sum_\alpha [T_\alpha^+, T_\alpha]$.

M) $M = \operatorname{Tr}(Z + B_0) - \operatorname{Tr}(\Delta)$ $(\Delta = d^+ d + d d^+)$.
S) $S = \sum_{v,j}(-1)^i Z_v \log h_v^j - \sum_{\alpha,j} \operatorname{Tr}(T_\alpha^{j+} T_\alpha^j) - \sum_{\alpha_j} \operatorname{Tr}[d^+, d]$.
F) $\mathcal{F} = \pi \operatorname{Deg} - \arg(Z + B_0)$

Theorem 2.2. *There is a structure of a DUY category on the derived category of representations of the quiver Q with the mass M, Kähler potential S, flow $\mathcal{F}$, etc, given by the formulas in the paragraph above.*

3. Some Applications

The theory of DUY categories paves the way for many applications. One application that we plan to pursue as part of this project is that of defining categorical analogues of Donaldson invariants. The basic idea is to replace the moduli space of stable bundles in Donaldson's work by the moduli space of semistable objects in a K-linear stable ∞-category $\mathcal{X}$, and the Betti cohomology of a variety by the periodic cyclic homology of $\mathcal{X}$. The main challenge in order complete this project will be to construct virtual fundamental classes for suitable moduli spaces of objects. The rest of the theory will be formally very similar to Donaldson's theory. However, the tremendous added flexibility afforded by working with categories instead of actual varieties (there are many more equivalence of categories), will allow us the apply this theory of categorical Donaldson invariants to questions of rationality in algebraic geometry that have resisted classical methods.

Let us consider the central charge $\int_X \operatorname{tr} e^{\left(\frac{i\omega}{\hbar}+F\right)}\Gamma$. For category $\mathrm{D}^{\mathrm{b}}(X)$, $\hbar \to 0$ gives $\frac{1}{n!}\left(w^{\dim X}\operatorname{rk}(E)\right) - \frac{i}{(n-1)!}w^{\dim+1}c_1(E) + \cdots$.

Conjecture 3.1. *Using the* $\operatorname{Ch}(U)$ *for* U *a universal bundle, we can define categorical Donaldson invariants as follows.*

U $\downarrow$ $H^2(Mod^\theta) \times H_2(X)$	U $\downarrow$ $H^*(Mod^\theta) \times HP_*(\mathrm{D}^{\mathrm{b}}(X))$

Here Mod^θ *is the moduli space of stable objects with* θ *slope.*
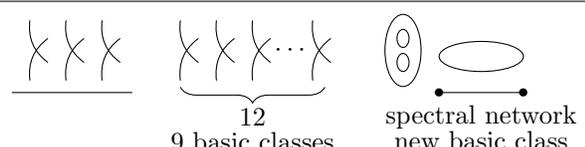
Donaldson invariants for $\mathrm{D}^{\mathrm{b}}(X)$ give us a way of studying semi-orthogonal decompositions. Consider category A_n.

Theorem 3.2. *The basic classes of* $Don(A_n)=\{generators\ of\ Orlov\ Spec(A_n)\}$.

This leads to a parallel between the theory of elliptic fibrations and Fano 3-folds of $Pic=1$.

Elliptic Fibrations	3-dim Fanos
$E_1 \xrightarrow{\substack{\text{rational} \\ \text{blow-ups}}} E_2$ surgeries Log transforms rational blow-downs	degeneration $X_1 \rightsquigarrow X'_{\text{tor}}$ $\downarrow$ bir degeneration $X_2 \rightsquigarrow X''_{\text{tor}}$

In case of Elliptic surfaces, the Donaldson invariants depend on the vanishing classes of blow-downs. Indeed we start with $\mathbb{P}^2$ blown up in 9 points - intersection of two smooth cubics in $\mathbb{P}^2$.

| Surfaces	$\mathbb{P}^2 \dashleftarrow \widehat{\mathbb{P}^2}_{p_1,\ldots,p_9} \xrightarrow{\substack{\text{Log } 2 \\ \text{Log } 3}} \text{Dolg } 2,3$ $t_1,\ldots,t_9$
Sheaf of Categories	12 9 basic classes spectral network new basic class

Consider the degeneration and regeneration of 3-dimensional Fanos.

$$X_1 \xrightarrow{deg} X'_{\text{tor}} \xrightarrow{bir} X'_{\text{tor}} \xleftarrow{bir} X_2 \qquad (3.1)$$

Conjecture 3.3. *The degeneration of MHS associated with degenerations $X_1 \xrightarrow{deg} X'_{tor}$, $X_2 \xrightarrow{deg} X''_{tor}$ of 3-dimensional Fano's leads to a connection $Don(X_1) = Don(X_2)$ ($Don(MHS)$).*

Conjecture 3.4. *For 3-dim Fanos X_1, X_2, $Pic(X_1)=Pic(X_2)$, $Don(MHS)$ is connected with Bardelli invariant $\psi(\alpha_1, \alpha_2)$.*

Idea of Proof. As before new basic classes appear after change of the monodromy in the sheaf categories.
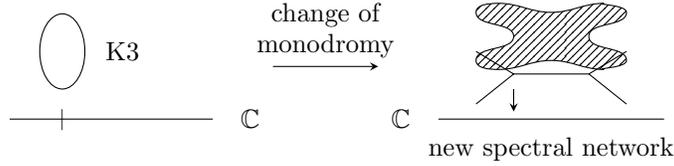
Change of monodromy leads to

- new spectral network.
- new moduli space of a new basic class.
- change of Bardelli form.

$\square$

We need to mention here that there is an issue with Wall crossing. In order to compare Donaldson invariants we need wall crossing formula for Moduli spaces of objects with wall crossing formula in Donaldson theory.

Conjecture 3.5. *In case of Dolgachev surface we have a wall crossing formula on the Donaldson invariants for moduli spaces of objects which in limit produces the classical Donaldson formula*

$$Don(C) = Don(C') + \sum_i e_i$$

In dim 4 we have also (4-dim cubic)

$$LG(X_1) \xrightarrow{\text{Log transform}} LG(X_2). \tag{3.2}$$

This Log transform changes the basic classes. In the case of X_1 being rational 4-dim cubic and X_2 being generic 4-dim cubic, we lose $\mathcal{O}_{K3}$ as a basic class. We arrive at several correspondences with classical Donaldson theory.

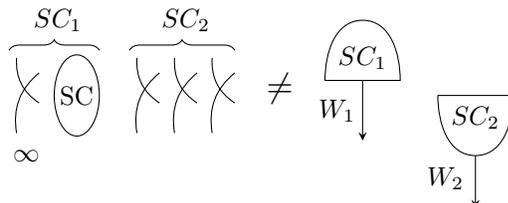| Classical Donaldson Theory	Categorical Donaldson Theory
$X = X_1 \#_C X_2$	Sheaf of categories is not connected with a potential
	Local sheaf of categories does not split in two sheaves which are not conncected with potentials

We recall two examples.

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(1) Dolgachev surface.

$$X = X_1 \# X_2$$
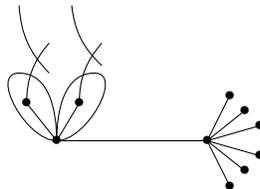
No function on these sheaves of categories

(2) 4-dim cubic.

$$\overbrace{\qquad}^{SC_1} \quad \overbrace{\qquad}^{SC_2} \qquad \neq$$

Obstructions to such splitting are the mod spaces of stable objects.

The following theorem could lead to a new approach to irrationality.

Theorem 3.6. *Let X be a rational manifold. Then $SC(X)$ the sheaf of categories associated with the LG mirror of X can be connected with a potential.*

We conjecture that the sheaves of categories $SC(X)$ are determined by moduli space of objects.

Conjecture 3.7. *The sheaves of categories $SC(X)$ are determined by moduli space of objects and sheaves on dg algebras on them.*

The Donaldson invariants (if properly defined) determine if $SC(X)$ are sheaves of categories given by a function.

Clearly constructing sheaves of categories not coming from potentials could have an important application in studying rationality questions. Before giving several procedures of constructing sheaves of categories $SC(X)$ not connecting with potentials we return to our basic examples the 2 dimensional LG model for the Dolgachev surface. We need to mention that similarly to Donaldson's treatment of Dolgachev surface we also encounter an issue with wall crossing. In order to compare Donaldson invariants we need wall crossing formulas for Moduli spaces of objects.

Conjecture 3.8. *In case of Dolgachev surface we have a wall crossing formula on the Donaldson invariants for moduli spaces of objects which in limit produces the classical Donaldson formula $Don(C) = Don(C') + \Sigma_i e_i$.*

Theorem 3.9. *The sheaves of categories $SC(X)$ associates with the 2 dimensional LG model for the Dolgachev surface is not connected with a potential.*

Indeed in this case the sheaf of categories does not come form the push forward of the structure but from the of a gerb. The 4 dimensional LG model for the Dolgachev surface produces a sheaf of categories which is connected with a potential. This observation suggests the following.

Theorem 3.10. *The following procedures could lead to sheaves of categories $SC(X)$ not connected with a potential:*

1) changing the structure sheaf of the initial sheaf of categories coming from a potential to to a gerb.

2) Taking a covering of a sheaf of categories.

3) Taking a part of a sheaf of categories.

This observations suggest that sheaves of categories coming from potentials play the role of simple 4 manifolds in Donaldson's theory and the procedure of splitting these sheaves is the analogue of the procedure of streching the neck.

Question: Consider the $SC(X)$ - the sheaf of categories of LG model of a Fano threefold X. Assume that $SC(X) = (SC_1, SC_2)$ i.e. we split $SC(X)$ to two sheaves of categories and one of them SC_2) is not connected with a potential. Can we then claim that X is not rational.

A positive answer to this question will build a categorical parallel to Voisin's theory of CH^0- trivial Fano varieties.